Lecture 4

6.4* - General Exponentials and Logarithms

Let $a>0$, then for any real number x:

$$a^x =$$

Notice that we must have $a>0$ because

Algebraic Properties

$a^{x+y} =$ $\qquad$ $a^{x-y} =$ $\qquad$ $(a^x)^y =$ $\qquad$ $(ab)^x =$

Ex: Simplify

$$\frac{(a^x)^2 a^{x^2+1}}{a^2}$$

Differentiation

$$\frac{d}{dx}\left(a^x\right) =$$

so, by the chain rule

$$\frac{d}{dx}\left(a^{g(x)}\right) =$$

Ex: Differentiate $k(x) = x^5 + 5^x$

Integration looks similar to before as well

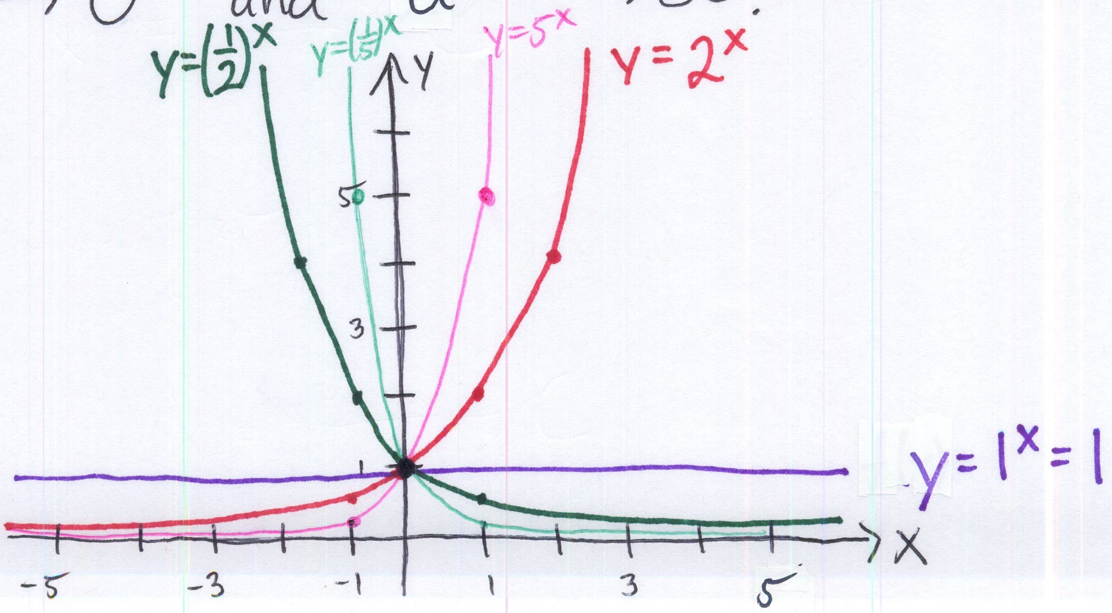
$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

and, by u-substitution

$$\int g'(x) a^{g(x)} \, dx = \frac{a^{g(x)}}{\ln a} + C$$

How does changing the value of a change the graph of $f(x) = a^x$?

- When $a=1$, $f(x) = 1^x = 1$, a horizontal line.
- For $a>1$, a^2, for example, increases as a increases. So, the graph becomes steeper. Also, $a^x \xrightarrow{x\to\infty} \infty$ and $a^x \xrightarrow{x\to -\infty} 0$.
- In the case $a<1$, let's write $b = \frac{1}{a}$. Then $b>1$ and $f(x) = a^x = \left(\frac{1}{b}\right)^x = b^{-x}$. So, this is like the previous case, but reflected about the y-axis. So as b increases, meaning that a decreases, the curve gets steeper. Finally $a^x \xrightarrow{x\to\infty} 0$ and $a^x \xrightarrow{x\to -\infty} \infty$.



The most general kind of exponential function is of the form $f(x)^{g(x)}$

We can differentiate this in two ways:-

- logarithmic differentiation
- writing $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$

Ex: Differentiate $y = (\sin x)^{\ln x}$

Ex: Find the limit $\lim\limits_{x \to 0^+} x^{-\ln x}$

one-to-one, and so has an inverse, which we call

$$f^{-1}(x) = \log_a x$$

(for example, $\log_e x = \ln x$)

We can interpret $\log_a x$ in terms of $\ln x$ as:

$$y = \log_a x \iff a^y = x \iff y \ln a = \ln a^y = \ln x \iff y = \frac{\ln x}{\ln a}$$

This is the change of base formula:

The usual rules for logarithms hold too:

$$\log_a 1 = 0 \qquad \log_a(xy) = \log_a x + \log_a y \qquad \log_a(x^r) = r \log_a x$$

From the change of base formula, we have:

$$\frac{d}{dx}(\log_a x) =$$

and the chain rule gives

$$\frac{d}{dx}(\log_a g(x)) =$$

Ex: Compute

$$\frac{d}{dx}\left(\log_5(xe^x)\right)$$

Let's make the number e more of a concrete thing:

Take the derivative of $f(x)=\ln x$ at $x=1$:

$$1=\frac{1}{1}=f'(1)=\lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x}=\lim_{x\to 0}\frac{\ln(1+x)}{x}$$

$$=\lim_{x\to 0}\ln\left[(1+x)^{1/x}\right]$$

Since e^x is a continuous function, we apply it to both sides and get:

$$e=e^1=e^{\lim_{x\to 0}\ln(1+x)^{1/x}}=\lim_{x\to 0}e^{\ln(1+x)^{1/x}}=\lim_{x\to 0}(1+x)^{1/x}$$

Using this limit, we can approximate e. First, notice that, with the substitution $n = \frac{1}{x}$:

$$e = \lim_{x \to 0} (1+x)^{1/x} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

So, plugging in larger values for n gives a better approximation of e:

$n=$	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.7181459...

The known value of e:

$e = 2.71828\ldots$

(e is an irrational number, in fact, it's what is called <u>transcendental</u> since it is not a zero of a polynomial with rational coefficients.)

Ex: Compute $\lim_{x \to 0} \left(1 + \frac{x}{3}\right)^{1/x}$